

# NON-DISSIPATIVE ELECTROMAGNETIC MEDIUM WITH A DOUBLE LIGHT CONE

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**ABSTRACT.** We study Maxwell's equations on a 4-manifold where the electromagnetic medium is modelled by an antisymmetric  $\binom{2}{2}$ -tensor with real coefficients. In this setting the *Fresnel surface* is a fourth order polynomial surface in each cotangent space that acts as a generalisation of the light cone determined by a Lorentz metric; the Fresnel surface parameterises electromagnetic wave-speeds as a function of direction. The contribution of this paper is a pointwise description of all electromagnetic medium tensors that satisfy the following conditions:

- (i)  $\kappa$  is invertible,
- (ii)  $\kappa$  is skewon-free,
- (iii)  $\kappa$  is *birefringent*, that is, the Fresnel surface of  $\kappa$  is the union of two distinct light cones.

We show that there are only three classes of mediums with these properties and give explicit expressions in local coordinates for each class.

We will study the *pre-metric* Maxwell's equations. In this setting Maxwell's equations are written on a 4-manifold  $N$  and the electromagnetic medium is described by an antisymmetric  $\binom{2}{2}$ -tensor  $\kappa$  on  $N$ . Then the electromagnetic medium  $\kappa$  determines a fourth order polynomial surface in each cotangent space called the *Fresnel surface*  $\mathcal{F}$  and it acts as a generalisation of the light cone determined by a Lorentz metric; the Fresnel surface parameterises wave-speeds as a function of direction [Rub02, HO03, PSW09]. At each point in spacetime  $N$ , the electromagnetic medium depends, in general, on 36 free components. In this work we assume that the medium is *skewon-free*. Then there are only 21 free components and such medium describe non-dissipative medium. For example, in skewon-free medium Poynting's theorem holds under suitable assumptions.

The above means that in the pre-metric setting we have two descriptions of electromagnetic medium: First, we have the  $\binom{2}{2}$ -tensor  $\kappa$  that contains the coefficients in Maxwell's equations. On the other hand, we also have the Fresnel surface  $\mathcal{F}$ , which describes the behaviour of a wavespeed for a propagating electromagnetic wave. If  $\kappa$  is known we can always compute  $\mathcal{F}$  by an explicit equation (see equation (10)). A more challenging question is to understand the converse dependence, or inverse problem: If the Fresnel surface  $\mathcal{F}|_p$  is known for some  $p \in N$ , what can we say about  $\kappa|_p$ ? Essentially this asks that if the behaviour of wave speed for an electromagnetic medium is known, what can we say about the medium? These questions are of theoretical interest, but also of practical interest as they relate to understanding measured data in engineering applications like traveltime tomography in anisotropic medium. We will here only study the problem at a point  $p \in N$  since the dependence will never be unique. For example, the Fresnel surface  $\mathcal{F}$  is always invariant under scalings and inversions of  $\kappa$  [HO03, Dah11a]. In general, these are not the only invariances, and for a general  $\kappa$  the relation between  $\kappa$  and  $\mathcal{F}$  does not seem to be very well understood.

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A natural first task is to characterise those mediums  $\kappa$  for which the Fresnel surface  $\mathcal{F}$  is the light cone of a Lorentz metric  $g$ . This question was raised in [OH99, OFR00]. A partial solution was given in [OFR00], and (in skewon-free mediums with real coefficients) the complete solution was given in [FB11]. The result is that if the Fresnel surface is a light cone, then  $\kappa$  is necessarily proportional to a Hodge star operator (plus, possibly, an axion component proportional to the identity). For an alternative proof, see [Dah11a], and for related results, see [OR02, LH04, Iti05].

The main contribution of this paper is Theorem 2.1. It gives a pointwise characterisation of all electromagnetic medium tensors  $\kappa$  with real coefficients such that

- (i)  $\kappa$  is invertible,
- (ii)  $\kappa$  is skewon-free,
- (iii)  $\kappa$  is *birefringent*, that is, the Fresnel surface of  $\kappa$  is the union of two distinct light cones for Lorentz metrics.

The first two assumptions imply that  $\kappa$  is essentially in one-to-one correspondence with an *area metric*. Area metrics also appear when studying the propagation of a photon in a vacuum with a first order correction from quantum electrodynamics [DH80, SWW10]. The Einstein field equations have also been generalised into equations where the unknown field is an area metric [PSW07]. For further examples, see [PSW09, SWW10] and for the differential geometry of area metrics, see [SW06, PSW07]. For Maxwell's equations, the interpretation of condition (iii) is that differently polarised waves can propagate with different wave speeds. In such medium one should expect that propagation of electromagnetic waves is determined by null-geodesics of two Lorentz metrics. A typical example of such medium is a uniaxial crystal. For partial results describing when the Fresnel surface factorises, see [RS11] and in 3 dimensions, see [Kac04, Dah06].

In Theorem 2.1 we show that there are only three medium classes with the above properties and we give explicit expressions in local coordinates for each class. Of these classes, the first is a generalisation of uniaxial medium and the last seems to be unphysical; heuristic arguments suggest that Maxwell's equations are not hyperbolic in the last class.

The main idea of the proof is as follows. We will use the normal form theorem for area metrics derived in [SWW10], which pointwise divides area metrics into 23 metaclasses and gives explicit expressions in local coordinates for each metaclass. This result was also used in [FB11], and by [SWW10] we only need to consider the first 7 metaclasses. For each of these metaclasses, the Fresnel surface can be written as  $\mathcal{F}|_p = \{\xi \in \mathbb{R}^4 : f(\xi) = 0\}$  for a homogeneous 4th order polynomial  $f: \mathbb{R}^4 \rightarrow \mathbb{R}$  with coefficients determined by  $\kappa|_p$ . Since  $\kappa|_p$  is birefringent,  $f$  factorises as

$$f(\xi) = f_+(\xi)f_-(\xi), \quad \xi \in \mathbb{R}^4$$

into homogeneous 2nd order polynomials  $f_{\pm}: \mathbb{R}^4 \rightarrow \mathbb{R}$ . By identifying coefficients we obtain a system of polynomial equations in coefficients of  $f$  and  $f_{\pm}$ . In the last step we eliminate the coefficients in  $f_{\pm}$  from these equations whence we obtain constraints on  $f$  (and hence on  $\kappa$ ) that must be satisfied when  $\kappa$  is birefringent. To eliminate variables we use the technique of *Gröbner bases*, which was also used in [Dah11a].

A limitation of Theorem 2.1 is that the explicit expression is only valid at a point. The reason for this is that the decomposition in [SWW10] essentially relies on the Jordan normal form theorem for matrices, which is unstable under perturbations.

Another limitation is that we do not allow for complex coefficients in  $\kappa$ . Therefore mediums like *chiral medium* are not included in the mediums in Theorem 2.1.

This paper relies on computations by computer algebra. For information about the Mathematica notebooks for these computations, please see the author's homepage.

## 1. MAXWELL'S EQUATIONS

By a *manifold*  $M$  we mean a second countable topological Hausdorff space that is locally homeomorphic to  $\mathbb{R}^n$  with  $C^\infty$ -smooth transition maps. All objects are assumed to be smooth and real where defined. Let  $TM$  and  $T^*M$  be the tangent and cotangent bundles, respectively. For  $k \geq 1$ , let  $\Lambda^k(M)$  be the set of antisymmetric  $k$ -covectors, so that  $\Lambda^1(N) = T^*N$ . Also, let  $\Omega_l^k(M)$  be  $\binom{k}{l}$ -tensors that are antisymmetric in their  $k$  upper indices and  $l$  lower indices. In particular, let  $\Omega^k(M)$  be the set of  $k$ -forms. Let  $C^\infty(M)$  be the set of functions. The Einstein summing convention is used throughout. When writing tensors in local coordinates we assume that the components satisfy the same symmetries as the tensor.

**1.1. Maxwell's equations on a 4-manifold.** On a 4-manifold  $N$ , Maxwell's equations read

$$(1) \quad dF = 0,$$

$$(2) \quad dG = j,$$

where  $d$  is the exterior derivative on  $N$ ,  $F, G \in \Omega^2(N)$  are the electromagnetic field variables, and  $j \in \Omega^3(N)$  is the source term. By an *electromagnetic medium* on  $N$  we mean a map

$$\kappa: \Omega^2(N) \rightarrow \Omega^2(N).$$

We then say that 2-forms  $F, G \in \Omega^2(N)$  solve *Maxwell's equations in medium  $\kappa$*  if  $F$  and  $G$  satisfy equations (1)–(2) and

$$(3) \quad G = \kappa(F).$$

Equation (3) is known as the *constitutive equation*. If  $\kappa$  is invertible, it follows that one can eliminate half of the free variables in Maxwell's equations (1)–(2). We assume that  $\kappa$  is linear and determined pointwise so that we can represent  $\kappa$  by an antisymmetric  $\binom{2}{2}$ -tensor  $\kappa \in \Omega_2^2(N)$ . If in coordinates  $\{x^i\}_{i=0}^3$  for  $N$  we have

$$(4) \quad \kappa = \frac{1}{2} \kappa_{lm}^{ij} dx^l \otimes dx^m \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

and  $F = F_{ij} dx^i \otimes dx^j$  and  $G = G_{ij} dx^i \otimes dx^j$ , then constitutive equation (3) reads

$$(5) \quad G_{ij} = \frac{1}{2} \kappa_{ij}^{rs} F_{rs}.$$

Then at each point on  $N$ , a general antisymmetric  $\binom{2}{2}$ -tensor  $\kappa$  depends on 36 free real components. In the main result of this paper (Theorem 2.1) we will assume that  $\kappa$  is *skewon-free*, that is,  $\kappa(u) \wedge v = u \wedge \kappa(v)$  for all  $u, v \in \Omega_2^2(N)$  whence  $\kappa$  has only 21 free components. Physically, such medium describe non-dissipative medium; if  $\kappa$  is time-independent, then Poynting's theorem holds under suitable assumptions [Dah10, Proposition 3.3]. Let us also note that if  $N$  is orientable, then invertible skewon-free mediums are essentially in a one-to-one correspondence with area metrics. See [SWW10] and [Dah11b, Proposition 2.4]. The medium is called *axion-free* if  $\text{trace } \kappa = 0$  [HO03].

By a *pseudo-Riemann metric* on a manifold  $M$  we mean a symmetric  $\binom{0}{2}$ -tensor  $g$  that is non-degenerate. If  $M$  is not connected we also assume that  $g$  has constant

signature. By a *Lorentz metric* we mean a pseudo-Riemann metric on a 4-manifold with signature  $(-+++)$  or  $(+---)$ . The *light cone* of a Lorentz metric is defined as

$$N(g) = \{\xi \in T_p^*(N) : g(\xi, \xi) = 0\}.$$

For  $p \in N$  we define  $N_p(g) = N(g) \cap T_p^*(N)$ .

If  $g$  is a pseudo-Riemann metric on a orientable 4-manifold  $N$ , then the Hodge operator of  $g$  induces a skewon-free  $\binom{2}{2}$ -tensor that we denote by  $\kappa = *_g$ . Moreover, if locally  $g = g_{ij}dx^i \otimes dx^j$ , and  $\kappa$  is written as in equation (4), then

$$(6) \quad \kappa_{rs}^{ij} = \sqrt{|g|} g^{ia} g^{jb} \varepsilon_{abrs},$$

where  $\det g = \det g_{ij}$ ,  $g^{ij}$  is the  $ij$ th entry of  $(g_{ij})^{-1}$ , and  $\varepsilon_{l_1 \dots l_n}$  is the *Levi-Civita permutation symbol*. We treat  $\varepsilon_{l_1 \dots l_n}$  as a purely combinatorial object (and not as a tensor density). Let also  $\varepsilon^{l_1 \dots l_4} = \varepsilon_{l_1 \dots l_4}$ .

**1.2. Representing  $\kappa$  as a  $6 \times 6$  matrix.** Let  $O$  be the ordered set of index pairs  $\{01, 02, 03, 23, 31, 12\}$ . If  $I \in O$ , we denote the individual indices by  $I_1$  and  $I_2$ . Say, if  $I = 31$  then  $I_2 = 1$ .

If  $\{x^i\}_{i=0}^3$  are local coordinates for a 4-manifold  $N$ , and  $J \in O$  we define  $dx^J = dx^{J_1} \wedge dx^{J_2}$ . A basis for  $\Omega^2(N)$  is given by  $\{dx^J : J \in O\}$ , that is,

$$(7) \quad \{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2\}.$$

This choice of basis follows [HO03, Section A.1.10] and [FB11]. If  $\kappa \in \Omega_2^2(N)$  is written as in equation (4) and  $J \in O$ , then

$$(8) \quad \kappa(dx^J) = \sum_{I \in O} \kappa_I^J dx^I, \quad J \in O,$$

where  $\kappa_I^J = \kappa_{I_1 I_2}^{J_1 J_2}$ . We will always use capital letters  $I, J, K, \dots$  to denote elements in  $O$ . Let  $b$  be the natural bijection  $b: O \rightarrow \{1, \dots, 6\}$ . Then we identify coefficients  $\{\kappa_I^J : I, J \in O\}$  for  $\kappa$  with the  $6 \times 6$  matrix  $A = (\kappa_I^J)_{IJ}$  defined as  $\kappa_I^J = A_{b(I)b(J)}$  for  $I, J \in O$  [Dah11b].

**1.3. Fresnel surface.** Let  $\kappa \in \Omega_2^2(N)$  on a 4-manifold  $N$ . If  $\kappa$  is locally given by equation (4) in coordinates  $\{x^i\}_{i=0}^3$ , let

$$\mathcal{G}_0^{ijkl} = \frac{1}{48} \kappa_{b_1 b_2}^{a_1 a_2} \kappa_{b_3 b_4}^{a_3 a_4} \kappa_{b_5 b_6}^{a_5 a_6} \varepsilon^{b_1 b_2 b_5 b_6} \varepsilon^{b_3 b_4 b_6 b_1} \varepsilon_{a_1 a_2 a_3 a_4}.$$

In overlapping coordinates  $\{\tilde{x}^i\}_{i=0}^3$ , these coefficients transform as

$$(9) \quad \tilde{\mathcal{G}}_0^{ijkl} = \det \left( \frac{\partial x^r}{\partial \tilde{x}^s} \right) \mathcal{G}_0^{abcd} \frac{\partial \tilde{x}^i}{\partial x^a} \frac{\partial \tilde{x}^j}{\partial x^b} \frac{\partial \tilde{x}^k}{\partial x^c} \frac{\partial \tilde{x}^l}{\partial x^d},$$

and components  $\mathcal{G}_0^{ijkl}$  define a tensor density  $\mathcal{G}_0$  on  $N$  of weight 1. The *Tamm-Rubilar tensor density* is the symmetric part of  $\mathcal{G}_0$  and we denote this tensor density by  $\mathcal{G}$  [Rub02, HO03, Dah11a]. In coordinates,  $\mathcal{G}^{ijkl} = \mathcal{G}_0^{(ijkl)}$ , where parenthesis indicate that indices  $ijkl$  are symmetrised with scaling  $1/4!$ . Using tensor density  $\mathcal{G}$ , the *Fresnel surface* at a point  $p \in N$  is defined as

$$(10) \quad \mathcal{F}|_p = \{\xi \in \Lambda_p^1(N) : \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = 0\}.$$

The Fresnel surface is a fundamental object when studying wave propagation in Maxwell's equations. It is clear that in each cotangent space, the Fresnel surface  $\mathcal{F}|_p$  is a fourth order polynomial surface, so it can have multiple sheets and singular points. There are various ways to derive the Fresnel surface; by studying a propagating weak singularity [OFR00, Rub02, HO03], using a geometric optics

[Iti09, Dah11a], or as the characteristic polynomial of the full Maxwell's equations [SWW10]. Classically, the Fresnel surface can be seen as the dispersion equation for a medium, so that it constrains possible wave speed(s) as a function of direction.

If  $\kappa = f *_g$  for a Lorentz metric  $g$  and a non-zero function  $f \in C^\infty(N)$ , then the Fresnel surface is the light cone of  $g$ . The converse is also true (assuming that  $\kappa$  is skewon-free and axion-free) [HO03, FB11, Dah11a]. The medium given by  $\kappa = f *_g$  is known as *non-birefringent medium*. For such medium propagation speed does not depend on polarisation.

**Definition 1.1.** Suppose  $N$  is a 4-manifold,  $\kappa \in \Omega_2^2(N)$  and  $\mathcal{F}$  is the Fresnel surface for  $\kappa$ . If  $p \in N$  we say that  $\mathcal{F}|_p$  *decomposes into a double light cone* if there exists Lorentz metrics  $g_+$  and  $g_-$  defined in a neighbourhood of  $p$  such that

$$(11) \quad \mathcal{F}|_p = N_p(g_+) \cup N_p(g_-)$$

and  $N_p(g_+) \neq N_p(g_-)$ . We then also say that  $\kappa|_p$  is *birefringent*.

We know that two Lorentz metrics are conformally related if and only if they have the same light cones [Ehr91]. The condition  $N_p(g_+) \neq N_p(g_-)$  thus only exclude non-birefringent mediums, which for skewon-free mediums are well understood (see above). When  $\mathcal{F}|_p$  decompose into a double light cone as in Definition 1.1 a physical interpretation is that the medium is *birefringent*. That is, differently polarised electromagnetic waves can propagate with different wave speeds. Common examples of such mediums are *uniaxial crystals* like calcite [BW80, Section 15.3].

To prove of the next four propositions we will need some terminology from algebraic geometry. If  $k = \mathbb{R}$  or  $k = \mathbb{C}$ , we denote by  $k[x_1, \dots, x_n]$  the ring of polynomials  $k^n \rightarrow k$  in variables  $x_1, \dots, x_n$ . Moreover, a non-constant polynomial  $f \in k[x_1, \dots, x_n]$  is *irreducible* if  $f = uv$  for  $u, v \in k[x_1, \dots, x_n]$  implies that  $u$  or  $v$  is a constant. For a polynomial  $r \in k[x_1, \dots, x_n]$ , let  $V(r) = \{x \in k^n : r(x) = 0\}$  be the *variety* induced by  $r$ , and let  $\langle r \rangle = \{fr : f \in k[x_1, \dots, x_n]\}$  be the *ideal generated by  $r$* . For what follows the necessary theory for manipulating these objects can, for example, be found in [CLO92].

The next proposition can be seen as a reformulation of the *Brill equations* that characterise when a homogeneous polynomial factorises into linear forms [Bri10].

**Proposition 1.2.** Suppose  $Q \in \mathbb{C}^{4 \times 4}$  is a symmetric non-zero matrix and  $f \in \mathbb{C}[\xi_0, \dots, \xi_3]$  is the polynomial  $f(\xi) = \xi^t \cdot Q \cdot \xi$  for  $\xi = (\xi_0, \dots, \xi_3) \in \mathbb{C}^4$ . Then  $f$  is irreducible in  $\mathbb{C}[\xi_0, \dots, \xi_3]$  if and only if  $\text{adj } Q \neq 0$ .

In Proposition 1.2,  $\text{adj } Q$  is the *adjugate matrix* of all cofactor expansions of  $Q$ , and  $\xi^t$  is the matrix transpose.

*Proof.* The result follows from the following three facts: First, if  $f = uv$  for polynomials  $u, v \in \mathbb{C}[\xi_0, \dots, \xi_3]$  then  $u$  and  $v$  are linear. (To see this, we know that  $Q(0) = 0$ , so we may assume that  $u(0) = 0$ . For a contradiction, suppose that  $v(0) \neq 0$ . Then  $df|_0 = 0$  implies that  $du|_0 = 0$ , but then  $u = 0$ , so  $Q = 0$ .) Second, by [Bri10, Example 2] polynomial  $f$  is a product of linear forms or  $f$  has a multiple factor if and only if  $G_f(\xi, \eta, \zeta) = 0$  for all  $\xi, \eta, \zeta \in \mathbb{C}^4$ . Here  $G_f$  is the *Gaeta covariant* defined as

$$G_f(\xi, \eta, \zeta) = -\frac{1}{2} \det \begin{pmatrix} 2f(\xi) & (Df)_\xi(\eta) & (Df)_\xi(\zeta) \\ (Df)_\xi(\eta) & 2f(\eta) & (Df)_\eta(\zeta) \\ (Df)_\xi(\zeta) & (Df)_\eta(\zeta) & 2f(\zeta) \end{pmatrix},$$

and  $(Df)_a(b) = \frac{d}{dt}(f(a+tb))|_{t=0}$  is the directional derivative. Third, by computer algebra we have  $\text{adj } Q = 0$  if and only if  $G_f = 0$ .  $\square$

In [Mon07] it is proven that the light cone of a Lorentz metric can not contain a vector subspace of dimension  $\geq 2$ . The next proposition generalise this result to double light cones. In the proof of Theorem 2.1 we will use this result to show that medium tensors in the last 16 metaclasses in the classification of [SWW10] can not decompose into a double light cone. In [SWW10] this property was used to show that these last metaclasses are neither hyperbolic, so in these metaclasses, Maxwell's equations are not well-posed.

**Proposition 1.3.** *Suppose  $g_{\pm}$  are Lorentz metrics on a 4-manifold  $N$ . If  $\Gamma \subset T_p N$  is a non-empty vector subspace such that  $\Gamma \subset N(g_+) \cup N(g_-)$ , then  $\dim \Gamma \leq 1$ .*

*Proof.* We may assume that  $\dim \Gamma \geq 1$ . Let us first prove the result in the special case that  $g_+$  and  $g_-$  are conformally related (after [Mon07, Proposition 2]). Let  $\{x^i\}_{i=0}^3$  be coordinates around  $p$  such that  $g_+|_p = \pm \text{diag}(-1, 1, 1, 1)$ , whence we can identify  $T_p M$  and  $\mathbb{R} \oplus \mathbb{R}^3$ . Let  $\pi$  be the Cartesian projection onto the first component. Then the restriction  $\pi|_{\Gamma} : \Gamma \rightarrow \mathbb{R}$  satisfies  $\ker \pi|_{\Gamma} = \{0\}$  and  $\dim \text{range } \pi|_{\Gamma} = 1$ , and the claim follows.

For general  $g_+$  and  $g_-$  let us show that  $\dim \Gamma \geq 2$  leads to a contradiction. If  $\dim \Gamma \geq 2$ , we can find linearly independent  $u, v \in \Gamma$  such that  $\text{span}\{u, v\} \subset N(g_+) \cup N(g_-)$ . We may further assume that  $u \in N(g_+)$ . Let

$$U = \{\theta \in \mathbb{R} : \cos \theta u + \sin \theta v \notin N(g_-)\}.$$

For  $w \in T^*N$  let us write  $\|w\|^2 = g_+(w, w)$ . If  $U$  is empty, then  $\text{span}\{u, v\} \subset N(g_-)$  and the result follows from the special case. Otherwise there exists a  $\theta_0 \in U$  so that  $\|\cos \theta u + \sin \theta v\|^2 = 0$  for all  $\theta$  in some neighbourhood  $I_0 \ni \theta_0$ . Differentiating gives

$$\frac{1}{2} (\|v\|^2 - \|u\|^2) \cdot \sin 2\theta + g_+(u, v) \cdot \cos 2\theta = 0, \quad \theta \in I_0.$$

By computing the Wronskian of  $\sin 2\theta, \cos 2\theta$ , it follows that  $0 = \|u\|^2 = \|v\|^2$  and  $g_+(u, v) = 0$ . Thus  $\text{span}\{u, v\} \subset N(g_+)$ , but this contradicts the special case.  $\square$

The next proposition gives the pointwise description of the Tamm-Rubilar tensor density at points  $p \in N$  where the the Fresnel surface decomposes into a double light cone. Let us emphasize that the result is pointwise. For example, in equation (12) the two sides have different transformation rules.

**Proposition 1.4.** *Suppose  $N$  is a 4-manifold,  $\kappa \in \Omega_2^2(N)$ , and the Fresnel surface of  $\kappa$  decomposes into a double light cone at  $p \in N$ . If  $\{x^i\}_{i=0}^3$  are coordinates around  $p$  and  $\mathcal{G}^{ijkl}$  and  $g_{\pm} = g_{\pm ij} dx^i \otimes dx^j$  are as in Definition 1.1, then*

$$(12) \quad \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = C (g_+^{ij} \xi_i \xi_j) (g_-^{kl} \xi_k \xi_l) \text{ at } p, \quad \{\xi_i\}_{i=0}^3 \in \mathbb{R}^4,$$

for some  $C \in \mathbb{R} \setminus \{0\}$ .

*Proof.* Let  $f_{\pm}, \gamma : \mathbb{C}^4 \rightarrow \mathbb{C}$  be polynomials  $f_{\pm}(\xi) = g_{\pm}^{ij} \xi_i \xi_j$  and  $\gamma(\xi) = \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l$  for  $\xi = (\xi_i)_{i=0}^3 \in \mathbb{C}^4$ . For ideals  $I_{\pm} = \langle f_{\pm} \rangle$  we then have  $\langle f_+ f_- \rangle = I_+ \cap I_-$  whence equation (11) implies that  $V(\langle \gamma \rangle) = V(I_+ \cap I_-)$  and passing to ideals gives

$$I(V(\langle \gamma \rangle)) = I(V(I_+ \cap I_-)).$$

The Strong Nullstellensatz implies that [CLO92, p. 175]

$$\langle \gamma \rangle \subset \sqrt{I_+} \cap \sqrt{I_-},$$

where  $\sqrt{I}$  be the *radical* of an ideal  $I$  and we used identities  $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$  and  $I \subset \sqrt{I}$  valid for any ideals  $I$  and  $J$  [CLO92, Proposition 16 in Section 4.3]. By Proposition 1.2,  $f_{\pm}$  are irreducible polynomials, so  $I_{\pm}$  are prime ideals and  $I_{\pm} = \sqrt{I_{\pm}}$ . Thus

$$\langle \gamma \rangle \subset \langle f_+ f_- \rangle$$

and  $\gamma = p f_+ f_-$  for some polynomial  $p: \mathbb{C}^4 \rightarrow \mathbb{C}$ . Computing the polynomial degree of both sides implies that  $p$  is a non-zero constant  $p = C \in \mathbb{C}$ . By Proposition 1.3, we can find a  $\xi \in \mathbb{R}^4$  so that  $\xi \notin V(f_+) \cup V(f_-) = V(\gamma)$  whence  $C \in \mathbb{R} \setminus \{0\}$ .  $\square$

For two Lorentz metrics  $g$  and  $h$  we know that their light cones  $N(g)$  and  $N(h)$  coincide if and only if  $g$  and  $h$  are conformally related [Ehr91]. The next proposition gives an analogous uniqueness result for Fresnel surfaces that decompose into a double light cone.

**Proposition 1.5.** *Suppose  $N$  is a 4-manifold,  $\kappa \in \Omega_2^2(N)$  and the Fresnel surface of  $\kappa$  decomposes into a double light cone at  $p \in N$ . Suppose furthermore that equation (11) holds for Lorentz metrics  $g_{\pm}$  and for Lorentz metrics  $h_{\pm}$ . Then there exists constants  $C_{\pm} \in \mathbb{R} \setminus \{0\}$  such that exactly one of the following conditions hold:  $g_{\pm} = C_{\pm} h_{\pm}$  at  $p$  or  $g_{\mp} = C_{\pm} h_{\pm}$  at  $p$ .*

*Proof.* The result follows by Propositions 1.2 and 1.4 and since any polynomial has a unique decomposition into irreducible factors [CLO92, Theorem 5 in Section 3.5].  $\square$

The next example shows that unique decomposition is not true when  $\mathcal{F}$  only decompose into second order surfaces.

**Example 1.6.** In coordinates  $\{x^i\}_{i=0}^3$  for  $\mathbb{R}^4$ , let  $\kappa$  be the skewon-free  $\binom{2}{2}$ -tensor determined by the  $6 \times 6$  matrix  $(\kappa_I^J)_{IJ} = \text{diag}(-1, 1, 0, -1, 1, 0)$ . Then  $\kappa$  has Fresnel surface

$$\mathcal{F} = \{\xi \in T^*\mathbb{R}^4 : \xi_0 \xi_1 \xi_2 \xi_3 = 0\}.$$

It is clear that  $\mathcal{F}$  has multiple factorisations into quadratic forms, and by Proposition 1.3,  $\mathcal{F}$  does not factorise into a double light cone.  $\square$

## 2. NON-DISSIPATIVE MEDIA WITH A DOUBLE LIGHT CONE

Theorem 2.1 below is the main result of this paper. To formulate the theorem we first need some terminology. Suppose  $L: V \rightarrow V$  is a linear map where  $V$  is a  $n$ -dimensional real vector space. If the matrix representation of  $L$  in some basis is  $A \in \mathbb{R}^{n \times n}$  and  $A$  is written using the Jordan normal form we say that  $L$  has *Segre type*  $[m_1 \cdots m_r \overline{k_1 k_1} \cdots \overline{k_s k_s}]$  when the blocks corresponding to real eigenvalues have dimensions  $m_1 \leq \cdots \leq m_r$  and the blocks corresponding to complex eigenvalues have dimensions  $2k_1 \leq \cdots \leq 2k_s$ . Moreover, by uniqueness of the Jordan normal form, the Segre type depends only on  $L$  and not on the basis. For a  $\binom{2}{2}$ -tensor  $\kappa$  on a 4-manifold, we define the Segre type of  $\kappa|_p$  as the Segre type of the linear map  $\Omega^2(N)|_p \rightarrow \Omega^2(N)|_p$  in basis (7). By counting how many ways a  $6 \times 6$  matrix can be decomposed into Jordan normal forms, it follows that there are only 23 Segre types for a  $\binom{2}{2}$ -tensor. A main result of [SWW10] are simple normal forms in local coordinates for each of these Segre types under the assumption that  $\kappa|_p$  is skewon-free and invertible. In the below proof we will use the restatement of this result in [Dah11b].

In the proof of Theorem 2.1 we will eliminate variables in systems of polynomial equations. Suppose  $V \subset \mathbb{C}^n$  is the solution set to polynomial equations  $f_1 = 0, \dots, f_N = 0$  where  $f_i \in \mathbb{C}[x_1, \dots, x_n]$ . If  $I$  is the ideal generated by  $f_1, \dots, f_N$ , the *elimination ideals* are the polynomial ideals defined as

$$I_k = I \cap \mathbb{C}[x_{k+1}, \dots, x_n], \quad k \in \{0, \dots, n-1\}.$$

Thus, if  $(x_1, \dots, x_n) \in V$  then  $p(x_{k+1}, \dots, x_n) = 0$  for any  $p \in I_k$ , and  $I_k$  contain polynomial consequences of the original equations that only depend on variables  $x_{k+1}, \dots, x_n$ . Using Gröbner basis, one can explicitly compute  $I_k$  [CLO92, Theorem 2 in Section 3.1]. In the proof of Theorem 2.1, we will use the built-in Mathematica routine `GroebnerBasis` for such computations.

**Theorem 2.1.** *Suppose  $N$  is a 4-manifold and  $\kappa \in \Omega_2^2(N)$ . Furthermore, suppose that at some  $p \in N$*

- (i)  $\kappa|_p$  has no skewon component,
- (ii)  $\kappa|_p$  is invertible as a linear map  $\Lambda_p^2(N) \rightarrow \Lambda_p^2(N)$ ,
- (iii) the Fresnel surface  $\mathcal{F}|_p$  factorises into a double light cone.

Then  $\kappa|_p$  must have Segre type  $[1\bar{1}1\bar{1}1\bar{1}]$ ,  $[2\bar{2}1\bar{1}]$  or  $[111\bar{1}1\bar{1}]$ .

- (i) **Metaclass I:** If  $\kappa|_p$  has Segre type  $[1\bar{1}1\bar{1}1\bar{1}]$ , there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -\beta_3 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & \alpha_3 \end{pmatrix}.$$

for some  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$  and  $\beta_1, \beta_2, \beta_3 > 0$ . For  $i \in \{1, 2, 3\}$  let

$$(13) \quad D_i = \frac{(\alpha_{i'} - \alpha_{i''})^2 + \beta_{i'}^2 + \beta_{i''}^2}{\beta_{i'}\beta_{i''}}$$

where  $i'$  and  $i''$  are defined such that  $\{i, i', i''\} = \{1, 2, 3\}$  and  $i' < i''$ .

Then, for exactly one  $i \in \{1, 2, 3\}$  we have

$$(14) \quad D_i = 2, \quad D_{i'} = D_{i''}$$

and equation (11) holds for Lorentz metrics

$$g_{\pm} = \text{diag} \left( 1, \frac{1}{2} \left( -D_1 \pm \sqrt{D_1^2 - 4} \right), \frac{1}{2} \left( -D_2 \pm \sqrt{D_2^2 - 4} \right), \frac{1}{2} \left( -D_3 \pm \sqrt{D_3^2 - 4} \right) \right)^{-1}.$$

- (ii) **Metaclass II:** If  $\kappa|_p$  has Segre type  $[2\bar{2}1\bar{1}]$ , If  $\kappa|_p$  is in Metaclass II, there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & -\beta_2 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & \alpha_2 \end{pmatrix}$$



where  $\alpha_1, \alpha_2 \in \mathbb{R}$  and  $\beta_1 > 0$ . Then  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2$ , and equation (11) holds for Lorentz metrics

$$g_{\pm} = \begin{pmatrix} \pm 1 & 0 & 0 & \beta_1 \\ 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}^{-1}.$$

(iii) **Metaclass IV:** If  $\kappa|_p$  has Segre type  $[11\bar{1}\bar{1}\bar{1}]$ , there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_4 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & 0 & 0 & \alpha_3 \end{pmatrix}$$

for some  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$  and  $\beta_1, \beta_2 > 0$ . Then  $\alpha_1 = \alpha_2$ ,  $\beta_1 = \beta_2$ ,  $\alpha_4 \neq 0$ ,  $\alpha_3^2 \neq \alpha_4^2$  and equation (11) holds for Lorentz metrics

$$g_{\pm} = \text{diag} \left( 1, \frac{1}{2} \left( -D_1 \pm \sqrt{D_1^2 + 4} \right), \frac{1}{2} \left( -D_1 \pm \sqrt{D_1^2 + 4} \right), -1 \right)^{-1},$$

where

$$(15) \quad D_1 = \frac{(\alpha_2 - \alpha_3)^2 + \beta_2^2 - \alpha_4^2}{\beta_2 \alpha_4}.$$

*Proof. Metaclass I.* The local expression for  $\kappa|_p$  is given by [Dah11a, Theorem 3.2]. Then the Tamm-Rubilar tensor density for  $\kappa|_p$  satisfies

$$(16) \quad C^{-1} \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = \xi_0^4 + \xi_1^4 + \xi_2^4 + \xi_3^4 - D_0 \xi_0 \xi_1 \xi_2 \xi_3 \\ + \sum_{i=1}^3 D_i (\xi_{i'}^2 \xi_{i''}^2 - \xi_0^2 \xi_i^2), \quad \xi \in \mathbb{R}^4,$$

where  $C = \beta_1 \beta_2 \beta_3$  and  $D_0$  is given explicitly in terms of  $\alpha_1, \dots, \beta_3$ , and implicitly  $D_0$  satisfies

$$(17) \quad D_0^2 = 4(4 + D_1 D_2 D_3 - D_1^2 - D_2^2 - D_3^2).$$

By Proposition 1.4, there are real symmetric matrices  $A = (A^{ij})_{i,j=0}^3$  and  $B = (B^{ij})_{i,j=0}^3$  such that

$$(18) \quad C^{-1} \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = (\xi^t \cdot A \cdot \xi) (\xi^t \cdot B \cdot \xi), \quad \xi \in \mathbb{R}^4.$$

Writing out these equations shows that  $A^{00} B^{00} = 1$ . Hence  $A^{00}$  is non-zero, and by rescaling  $A$  and  $B$ , we may assume that  $A^{00} = 1$ . This substitution simplifies the equations so that by polynomial substitutions we may eliminate all variables in  $B$  and variable  $D_0$ . This results in a system of polynomial equations that only involve  $D_1, D_2, D_3$  and the variables in  $A$ . Further eliminating the variables in  $A$  using a Gröbner basis, gives constraints on  $D_1, D_2, D_3$ . By equation (13) we know that  $D_1, D_2, D_3 \geq 2$ , whence these constraints imply that there exists a unique  $i \in \{1, 2, 3\}$  such that conditions (14) hold. (To see that  $i$  is unique it suffices to note that if  $D_i = D_j = 2$  for  $i \neq j$  then  $D_1 = D_2 = D_3 = 2$  whence  $D_0 = 0$  and equation (18) holds for the single light cone  $A = B = g_0^{-1}$  with  $g_0 = \text{diag}\{-1, 1, 1, 1\}$ . By Proposition 1.4 and unique decomposition into irreducible factors this gives a contradiction.) Equations (14) and (17) imply that  $D_0 = 0$ . The result follows

since equation (18) holds with  $A = g_+^{-1}$  and  $B = g_-^{-1}$ , where  $g_{\pm}$  are the matrices in the theorem formulation.

**Metaclass II.** As in Metaclass I, there are matrices  $A$  and  $B$  such that the Fresnel polynomial satisfies equation (18) (with  $C = 1$ ). As in Metaclass I we can eliminate variables in  $B$ . Further eliminating all variables in  $A$  by a Gröbner basis implies that  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2$ . Then a direct computation shows that equation (18) holds with  $A = g_+^{-1}$ ,  $B = g_-^{-1}$  and  $C = \beta_1$ . Computer algebra shows that  $g_{\pm}$  have Lorentz signatures.

**Metaclass III.** If  $\kappa|_p$  is in Metaclass III, there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha_1 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & 1 \\ 0 & 0 & 1 & -\beta_1 & \alpha_1 & 0 \\ 0 & 1 & \beta_1 & 0 & 0 & \alpha_1 \end{pmatrix}$$

where  $\alpha_1 \in \mathbb{R}$  and  $\beta_1 > 0$ . Decomposing the Fresnel polynomial as in equation (18) (with  $C = 1$ ) gives a system of polynomial equations for the variables in  $A$ ,  $B$  and  $\kappa|_p$ . Computing the Gröbner basis for these equations implies that  $\beta_1 = 0$ . Thus  $\kappa|_p$  can not be in Metaclass III.

**Metaclass IV.** Let us first note that  $\alpha_4 \neq 0$  since otherwise  $\text{span}\{dx^1|_p, dx^2|_p\} \subset \mathcal{F}|_p$  which is not possible by Proposition 1.3. Then the Tamm-Rubilar tensor density satisfies

$$\begin{aligned} C^{-1} \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l &= \xi_0^4 - \xi_1^4 - \xi_2^4 + \xi_3^4 + D_0 \xi_0 \xi_1 \xi_2 \xi_3 \\ &\quad + D_1 (\xi_2^2 \xi_3^2 - \xi_0^2 \xi_1^2) + D_2 (\xi_1^2 \xi_3^2 - \xi_0^2 \xi_2^2) + D_3 (-\xi_1^2 \xi_2^2 - \xi_0^2 \xi_3^2), \end{aligned}$$

where  $C = \beta_1 \beta_2 \alpha_4$ ,  $D_0$  is determined explicitly in terms of  $\alpha_1, \dots, \beta_2$ ,  $D_1$  is defined in equation (15),  $D_3 \geq 2$  is defined in equation (13) and

$$D_2 = \frac{(\alpha_1 - \alpha_3)^2 + \beta_1^2 - \alpha_4^2}{\beta_1 \alpha_4}.$$

By decomposing and eliminating variables as in Metaclass I, it follows that that  $D_0 = 0$  and  $D_3 = 2$ . Thus we have proven that  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2$  whence  $D_1 = D_2$  and equation (18) holds with  $A = g_+^{-1}$ ,  $B = g_-^{-1}$  and  $C$  as above. Moreover,  $g_{\pm}$  both have Lorentz signatures. Condition  $\alpha_3^2 \neq \alpha_4^2$  follows since  $\det \kappa|_p \neq 0$ .

**Metaclass V.** If  $\kappa|_p$  is in Metaclass V, there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 & 0 \\ 1 & 0 & 0 & -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_2 \end{pmatrix}$$

where  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$  and  $\beta_1 > 0$ . We may assume that  $\alpha_3 \neq 0$ , since otherwise  $\text{span}\{dx^i|_p\}_{i=1}^3 \subset \mathcal{F}|_p$ . Decomposing and eliminating variables as in Metaclass I gives that  $\beta_1$  is purely complex. Thus  $\kappa|_p$  can not be in Metaclass V.

**Metaclass VI.** If  $\kappa|_p$  is in Metaclass VI, there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_5 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_5 & 0 & 0 & \alpha_3 \end{pmatrix}$$

for some  $\alpha_1, \dots, \alpha_5 \in \mathbb{R}$  and  $\beta_1 > 0$ . We may assume that  $\alpha_4$  and  $\alpha_5$  are non-zero since otherwise  $\text{span}\{dx^i, dx^2\} \subset \mathcal{F}|_p$  for some  $i \in \{0, 1\}$  as in Metaclass IV. Then the Tamm-Rubilar tensor density satisfies

$$\begin{aligned} C^{-1} \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l &= \xi_0^4 + \xi_1^4 - \xi_2^4 - \xi_3^4 + D_0 \xi_0 \xi_1 \xi_2 \xi_3 \\ &\quad + D_1 (\xi_2^2 \xi_3^2 - \xi_0^2 \xi_1^2) - D_2 (\xi_1^2 \xi_3^2 + \xi_0^2 \xi_2^2) - D_3 (\xi_1^2 \xi_2^2 + \xi_0^2 \xi_3^2), \end{aligned}$$

where  $C = \beta_1 \alpha_4 \alpha_5$  and  $D_0, D_1, D_2, D_3 \in \mathbb{R}$  are defined in terms of  $\alpha_i$  and  $\beta_1$ . By decomposing the Fresnel tensor as in equation (18) and eliminating variables using a Gröbner basis, it follows that there exists a  $\sigma \in \{\pm 1\}$  such that

$$D_0 = 0, \quad D_1 = \sigma 2, \quad D_2 = -\sigma D_3,$$

and moreover, equation (18) holds for  $A = g_+^{-1}$ ,  $B = g_-^{-1}$  and  $C$  as above, where

$$g_{\pm} = \text{diag} \left( 1, -\sigma, \frac{1}{2} \left( \sigma D_3 \pm \sqrt{D_3^2 + 4} \right), \frac{1}{2} \left( -D_3 \mp \sigma \sqrt{D_3^2 + 4} \right) \right)^{-1}.$$

Since  $g_+$  does not have a Lorentz signature for any  $\sigma \in \{\pm 1\}$  and  $D_3 \in \mathbb{R}$ , Proposition 1.4 and unique factorisation imply that  $\kappa|_p$  can not be in Metaclass VI.

**Metaclass VII.** If  $\kappa|_p$  is in Metaclass VII, there are coordinates  $\{x^i\}_{i=0}^3$  around  $p$  such that

$$(\kappa_I^J)_{IJ} = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \alpha_6 \\ \alpha_4 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_5 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_6 & 0 & 0 & \alpha_3 \end{pmatrix}$$

for some  $\alpha_1, \dots, \alpha_6 \in \mathbb{R}$ . We may assume that  $\alpha_4, \alpha_5, \alpha_6 \neq 0$  since otherwise  $\text{span}\{dx^i|_p, dx^j|_p\} \subset \mathcal{F}|_p$  for some  $i, j \in \{0, 1, 2\}$  as in Metaclass IV. Then the Tamm-Rubilar tensor density satisfies

$$C^{-1} \mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = \xi_0^4 + \xi_1^4 + \xi_2^4 + \xi_3^4 + D_0 \xi_0 \xi_1 \xi_2 \xi_3 - \sum_{i=1}^3 D_i (\xi_i^2 \xi_{i'}^2 + \xi_0^2 \xi_i^2),$$

where  $C = \alpha_4 \alpha_5 \alpha_6$ , constants  $D_1, D_2, D_3 \in \mathbb{R}$  are given by

$$(19) \quad D_1 = \frac{(\alpha_2 - \alpha_3)^2 - \alpha_5^2 - \alpha_6^2}{\alpha_5 \alpha_6},$$

$$(20) \quad D_2 = \frac{(\alpha_1 - \alpha_3)^2 - \alpha_4^2 - \alpha_6^2}{\alpha_4 \alpha_6},$$

$$(21) \quad D_3 = \frac{(\alpha_1 - \alpha_2)^2 - \alpha_4^2 - \alpha_5^2}{\alpha_4 \alpha_5},$$

and  $D_0 \in \mathbb{R}$  is given explicitly in terms of  $\alpha_1, \dots, \beta_3$ , and implicitly  $D_0$  satisfies

$$(22) \quad D_0^2 = 4(-4 + D_1 D_2 D_3 + D_1^2 + D_2^2 + D_3^2).$$

Decomposing the Tamm-Rubilar tensor density as in equation (18) and eliminating variables using a Gröbner basis, gives polynomial equations for  $D_0, D_1, D_2, D_3$ . Let us consider the cases  $D_0 = 0$  and  $D_0 \neq 0$  separately. If  $D_0 = 0$ , there exists an  $i \in \{1, 2, 3\}$  and a  $\sigma \in \{\pm 1\}$  such that

$$(23) \quad D_0 = 0, \quad D_i = -\sigma 2, \quad D_{i'} = \sigma D_{i''},$$

where the last condition is a consequence of equation (22). Suppose  $i = 1$ . Then Proposition 1.4 implies that for some invertible symmetric matrices  $A, B \in \mathbb{R}^{4 \times 4}$  with Lorentz signatures we have

$$(24) \quad (\xi^t \cdot A \cdot \xi) (\xi^t \cdot B \cdot \xi) = (\xi^t \cdot L_+ \cdot \xi) (\xi^t \cdot L_- \cdot \xi), \quad \xi \in \mathbb{C}^4,$$

where matrices  $L_{\pm} \in \mathbb{C}^{4 \times 4}$  are defined as

$$(25) \quad L_{\pm} = \text{diag} \left( 1, \sigma, \frac{1}{2} \left( -D_2 \pm \sqrt{D_2^2 - 4} \right), \frac{\sigma}{2} \left( -D_2 \pm \sqrt{D_2^2 - 4} \right) \right).$$

Since  $L_{\pm}$  are invertible, equation (24), Proposition 1.2 and unique factorisation imply that  $L_{\pm}$  are real and have Lorentz signatures. Thus  $|D_2| \geq 2$  and  $\det L_{\pm} < 0$ , but this contradicts equation (25), which implies that

$$\det L_{\pm} = \frac{1}{4} \left( -D_2 \pm \sqrt{D_2^2 - 4} \right)^2 > 0.$$

A similar analysis for  $i = 2, 3$  shows that the case  $D_0 = 0$  is not possible. If  $D_0 \neq 0$  it follows that there exists  $\sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}$  and distinct  $i, j, k \in \{1, 2, 3\}$  such that

$$(26) \quad D_0 \neq 0, \quad D_i = \sigma_1 2, \quad D_j = \sigma_2 2, \quad D_k = \frac{1}{2} (-4\sigma_1 \sigma_2 + \sigma_3 D_0),$$

where the last equation follows from equation (22). If  $(i, j) = (1, 2)$  then  $k = 3$  and Proposition 1.4 implies that for some invertible symmetric matrices  $A, B \in \mathbb{R}^{4 \times 4}$  with Lorentz signatures, equation (24) holds for matrices  $L_{\pm} \in \mathbb{C}^{4 \times 4}$  defined as

$$(27) \quad L_{\pm} = \begin{pmatrix} 1 & 0 & 0 & \pm \frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} \\ 0 & -\sigma_1 & \mp \frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & 0 \\ 0 & \mp \frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & -\sigma_2 & 0 \\ \pm \frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & 0 & 0 & \sigma_1 \sigma_2 \end{pmatrix}.$$

Since both sides in equation (24) should decompose into the same number of irreducible factors, it follows that  $\xi^t \cdot L_{\pm} \cdot \xi$  are irreducible in  $\mathbb{C}[\xi_0, \dots, \xi_3]$ . Thus equation (24) and unique factorisation imply that  $L_{\pm}$  are real and have Lorentz signatures, so  $\det L_{\pm} < 0$ . However, this contradicts equation (27) which implies that

$$\det L_{\pm} = \left( \frac{1}{8} D_0 - \sigma_1 \sigma_2 \sigma_3 \right)^2 \geq 0.$$

The cases  $(i, j) = (1, 3), (2, 3)$  are excluded by the same argument by using metrics

$$L_{\pm} = \begin{pmatrix} 1 & 0 & \pm \frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & 0 \\ 0 & -\sigma_1 & 0 & \mp \frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} \\ \pm \frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & 0 & \sigma_1\sigma_2 & 0 \\ 0 & \mp \frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & 0 & -\sigma_2 \end{pmatrix},$$

$$L_{\pm} = \begin{pmatrix} 1 & \pm \frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & 0 & 0 \\ \pm \frac{\sqrt{D_0}}{\sqrt{8}\sqrt{\sigma_3}} & \sigma_1\sigma_2 & 0 & 0 \\ 0 & 0 & -\sigma_1 & \mp \frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} \\ 0 & 0 & \mp \frac{\sqrt{D_0}\sqrt{\sigma_3}}{\sqrt{8}} & -\sigma_2 \end{pmatrix},$$

respectively. Thus  $\kappa|_p$  can not be in metaclasses VII.

**Metaclasses VIII—XXIII.** (Following [SWW10, Lemma 5.1].) Let  $A = (\kappa_I^J)_{IJ}$  be the  $6 \times 6$  matrix that represents  $\kappa|_p$  in some coordinates  $\{x^i\}_{i=0}^3$  around  $p$ . Then the Jordan normal form of  $A$  has a block of dimension  $d \in \{2, \dots, 6\}$  that corresponds to a real eigenvalue  $\lambda \in \mathbb{R} \setminus \{0\}$ . By considering unit vectors in the normal basis, we can find non-zero  $e_1, e_2 \in \Lambda^2(N)|_p$  so that  $\kappa(e_1) = \lambda e_1$  and  $\kappa(e_2) = \lambda e_2 + e_1$ . Writing out  $\kappa(e_1) \wedge e_2 = e_1 \wedge \kappa(e_2)$  implies that  $e_1 \wedge e_2 = 0$ , so  $e_1 = \eta_1 \wedge \eta_2$  for some linearly independent  $\eta_1, \eta_2 \in \Lambda^1(N)|_p$  [Coh05, p. 184]. Let  $W = \text{span}\{\eta_1, \eta_2\}$ . For all  $\xi \in W$  we then have

$$W \subset \{\alpha \in \Lambda^1(N)|_p : \xi \wedge \kappa(\xi \wedge \alpha) = 0\},$$

whence Theorem 3.3 in [Dah11a] implies that  $W \subset \mathcal{F}|_p$  and Proposition 1.3 implies that  $\kappa|_p$  can not be in metaclasses VIII–XXIII.  $\square$

In the proof of Theorem 2.1 the assumption that  $\kappa$  is invertible is only used to show that  $\alpha_3^2 \neq \alpha_4^2$  in Metaclass IV and to exclude Metaclasses VIII—XXIII. It would be interesting to see if these last metaclasses can be excluded also for non-invertible  $\kappa$  by other arguments. Regarding this question it should be emphasized that here  $\kappa$  is real. For complex coefficients  $\kappa$ , the setting becomes more involved. For example, in [Dah11a, Example 5.3] it is shown that for complex  $\kappa$  the Fresnel surface can be a single light cone even if  $\kappa$  is not invertible. Also, *chiral medium* would be an example of a medium with complex coefficients in  $\kappa$  and with a double light cone. In chiral medium right and left hand circularly polarised plane waves propagate with different wavespeeds. Let us note that if we set  $\xi_0 = 0$  in equation (16) we obtain the ternary quartic studied in [Tho16] and for this polynomial,  $D_0^2$  in equation (17) is one of the factors in the discriminant.

Let us make some comments regarding the three mediums derived in Theorem 2.1. A first observation is that for each Metaclass in Theorem 2.1, the light cones are parameterised by only one parameter:  $D_{i'} = D_{i''} \geq 2$  in Metaclass I,  $\beta_1 > 0$  in Metaclass II, and  $D_1 \in \mathbb{R}$  in Metaclass IV. Let consider each metaclass under the assumption that Theorem 2.1 holds.

**Metaclass I.** In Metaclass I we assume that conditions (14) holds for only one  $i \in \{1, 2, 3\}$ . In terms of  $\alpha_1, \dots, \beta_3$  conditions (14) are equivalent to

$$\alpha_{i'} = \alpha_{i''}, \quad \beta_{i'} = \beta_{i''}.$$

When  $\alpha_1 = \alpha_2 = \alpha_3 = 0$  this medium reduces to a uniaxial medium, where wave propagation is well understood.

Let us also note that if  $D_i = D_j = 2$  for two distinct  $i, j \in \{1, 2, 3\}$ , then  $\alpha_1 = \alpha_2 = \alpha_3$  and  $\beta_1 = \beta_2 = \beta_3$ . Then  $\kappa|_p = -\beta_1 *g + \alpha_1 \text{Id}$  for the locally defined Lorentz metric  $g = \text{diag}(-1, 1, 1, 1)$  and the Fresnel surface is the single light cone

$$\mathcal{F}|_p = N_p(g).$$

**Metaclass II.** For Metaclass II, the Fresnel polynomial  $\mathcal{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$  is a function of  $\xi_0, \xi_1^2 + \xi_2^2, \xi_3$ . It is therefore motivated to project  $\mathcal{F}|_p$  onto  $\xi_1 = 0$ , and we can plot  $\mathcal{F}|_p$  as a surface in  $\mathbb{R}^3$ . Figure 1 show this projection for three different values of  $\beta_1$ . From the figures (or from metrics  $g_{\pm}$ ) we see that the light cones  $N(g_{\pm})$  coincide in the limit  $\beta_1 \rightarrow \infty$ .

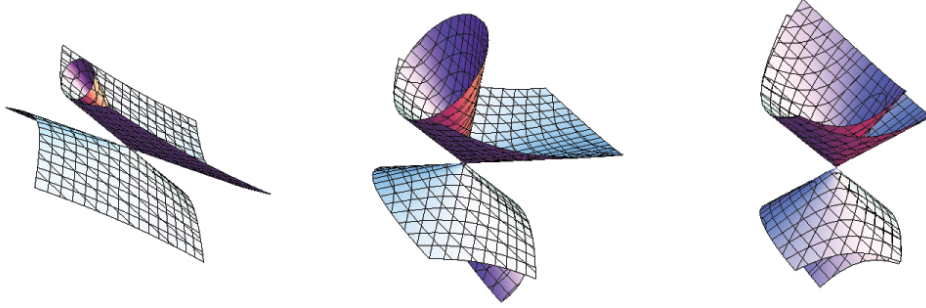


FIGURE 1. Projection into  $\mathbb{R}^3$  of Fresnel surfaces in Metaclass II for  $\beta_1 = 0.2$  (left),  $\beta_1 = 0.8$  and  $\beta_1 = 5$  (right).

By a coordinate transformation we can put the local representation of  $\kappa|_p$  into a more symmetric form. Let  $\{\tilde{x}^i\}_{i=0}^3$  be coordinates defined as  $\tilde{x}^i = \sum_{j=0}^3 L_{ij}x^j$  where  $L = (L_{ij})$  is the Jacobian matrix

$$L = \begin{pmatrix} 0 & 0 & \frac{1}{2\beta_1}(1-w) & \frac{1}{2\beta_1}(1+w) \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1},$$

where  $w = \sqrt{1 + 4\beta_1^2}$ . The motivation for these coordinates is that they diagonalize  $g_+$ . Then the  $6 \times 6$  matrix  $(\tilde{\kappa}_I^J)_{IJ}$  that represents  $\kappa|_p$  in  $\{\tilde{x}^i\}_{i=0}^3$  coordinates is

$$\alpha_1 \text{Id} + \frac{1}{w} \begin{pmatrix} 0 & 0 & 0 & \beta_1^2 & 0 & 0 \\ 0 & \beta_1 & -\beta_1 & 0 & \beta_1(-1+w) & -\beta_1 \\ 0 & \beta_1 & -\beta_1 & 0 & -\beta_1 & -\beta_1(1+w) \\ -w^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_1(1+w) & \beta_1 & 0 & \beta_1 & \beta_1 \\ 0 & \beta_1 & \beta_1(-1+w) & 0 & -\beta_1 & -\beta_1 \end{pmatrix}.$$

**2.1. Metaclass IV.** For Metaclass IV, the Fresnel polynomial is also a function of  $\xi_0, \xi_1^2 + \xi_2^2, \xi_3$ . We may therefore visualize the Fresnel surface in the same way as in Metaclass II. See Figure 2.

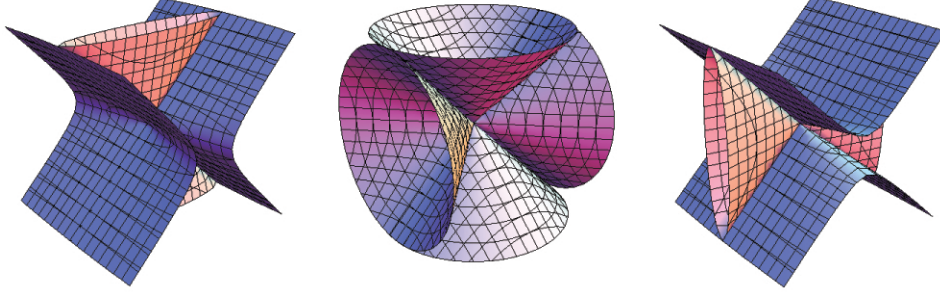


FIGURE 2. Projection into  $\mathbb{R}^3$  of Fresnel surfaces in Metaclass IV for  $D_1 = -25$  (left),  $D_1 = 0$  and  $D_1 = 25$  (right).

Suppose  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ . If we treat  $x^0$  as time and  $\{x^i\}_{i=1}^3$  as space coordinates, and write Maxwell's equations using vector fields  $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ , then  $\kappa|_p$  represents medium

$$\begin{aligned}\mathbf{D} &= -\text{diag}(\beta_1, \beta_1, \alpha_4) \cdot \mathbf{E}, \\ \mathbf{B} &= -\text{diag}(\beta_1, \beta_1, -\alpha_4)^{-1} \cdot \mathbf{H}.\end{aligned}$$

For an electromagnetic medium to be physically relevant, the Fresnel polynomial  $\mathcal{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$  should be *hyperbolic polynomial* [SWW10]. This is a necessary condition for Maxwell's equations to form a predictive theory, that is, a necessary condition for Maxwell's equations to be solvable forward in time. The above 3D projections and the argument in [SWW10] suggests that Metaclass II is hyperbolic for all  $\beta_1 > 0$  while Metaclass IV is never hyperbolic for any  $D_1 \in \mathbb{R}$ . This is also supported by some numerical tests.

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